Admissibility
Free algebra

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Free algebras

Spoiler

Quasi-identities valid in free algebras correspond to admissible rules.

Free algebras

Let $\mathcal{C}$ be a class of algebras in the same language. Let X be a set of variables. An algebra $\boldsymbol{Fr}(X)$ is **free over X for $\mathcal{C}$**, and **of rank** $\mathrm{card}(X)$ if

- there is an embedding $e\colon X \to \boldsymbol{Fr}(X)$
- (*Universal property*) for every mapping $v\colon X \to A$, $\boldsymbol{A} \in \mathcal{C}$ there exists a unique homomorphism $\bar{v}\colon \boldsymbol{Fr}(X) \to \boldsymbol{A}$ such that $v = \bar{f} \circ e$.

$$\begin{tikzcd}
 & X \arrow[dl, hook', "e"'] \arrow[dr, "v"] & \\
\boldsymbol{Fr}(X) \arrow[rr, dashed, "\exists! \bar{v}"'] & & \boldsymbol{A}
\end{tikzcd}$$

$\boldsymbol{Fr}(X)$ is free **in** $\mathcal{C}$ if it belongs to $\mathcal{C}$.

Definition

Let $\mathcal{C}$ be a class of algebras in the same language. Let X be a set of variables. An algebra $\boldsymbol{Fr}(X)$ is **free over X for $\mathcal{C}$**, and **of rank** $\mathrm{card}(X)$ if

$$\begin{tikzcd} & X \arrow[dl, hook', "e"'] \arrow[dr, "v"] & \\ \boldsymbol{Fr}(X) \arrow[rr, dashed, "\exists! \bar{v}"'] & & \boldsymbol{A} \end{tikzcd}$$

Remarks:

Often e is *hidden* in a sense that $e(x)$ is identified with x. One may think about e as inclusion. Then $\bar{v}$ is a homomorphic extension of v.

When $X = \{x_1, \ldots, x_n\}$ we may write $\boldsymbol{Fr}(x_1, \ldots, x_n)$, or $\boldsymbol{Fr}(\bar{x})$, or even $\boldsymbol{Fr}(n)$. $\boldsymbol{Fr}$ denotes $\boldsymbol{Fr}(x_1, x_2, \ldots)$.

First example of a free algebra

The algebra $\boldsymbol{Form}(X)$ of all formulas in a fixed language $\mathcal{L}$ and with all variables in X is free over X for the class of all algebras in the language $\mathcal{L}$.

It is quite common to identify the mapping $v\colon X \to A$ and the homomorphism $\bar{v}\colon \boldsymbol{Form}(X) \to \boldsymbol{A}$. Both are called a **valuation**. The bar $\bar{\ }$ is often omitted.

If $\boldsymbol{Fr}(X)$ is free for $\mathcal{C}$ and $\mathcal{D} \subseteq \mathcal{C}$ then $\boldsymbol{Fr}(X)$ is free for $\mathcal{D}$.
In particular, $\boldsymbol{Form}(X)$ is free for any class of algebras in a fixed language.

$$\begin{tikzcd} & X \arrow[dl, hook, "e"'] \arrow[dr, "v"] & \\ \boldsymbol{Fr}(X) \arrow[rr, dashed, "\exists! \bar{v}"'] & & \boldsymbol{A} \end{tikzcd}$$

Definition

Let $\mathcal{C}$ be a class of algebras in the same language. Let X be a set of variables. An algebra $\boldsymbol{Fr}(X)$ is **free over X in $\mathcal{C}$** if it is free over X for $\mathcal{C}$ and $\boldsymbol{Fr}(X) \in \mathcal{C}$.
In such a case we rather write $\boldsymbol{Fr}_{\mathcal{C}}(X)$.

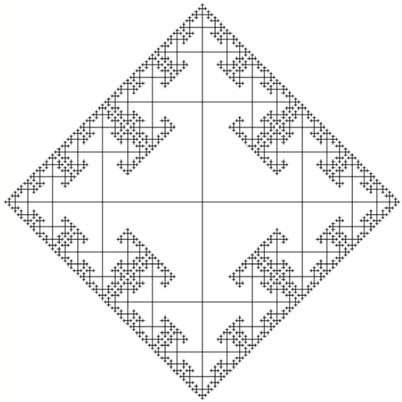
If $\mathcal{C}$ is a class of **** algebras and $\boldsymbol{Fr}(X)$ is free in $\mathcal{C}$, we say that $\boldsymbol{Fr}_{\mathbb{C}}(X)$ is a free **** algebra.

- $\mathbb{Z}^+$ is a free commutative semigroup over $\{1\}$;
- $\mathbb{N}$ is a free commutative monoid over $\{1\}$;
- $\mathbb{N}^k$ is a free commutative monoid over $\{(1,0,\ldots,0),\ldots,\{(0,\ldots,0,1)\}$;
- singleton $\{x\}$ is a free semilattice over $\{x\}$;
- $(\{0,1\}^n,\vee)$ is a free semilattice over $\{(1,0,\ldots,0),\ldots,\{(0,\ldots,0,1)\}$.

Exercise
Show that a free semillatice is a homomorphic image of a free commutative semigroup.

Exercise
Let $\mathcal{C}$ be a subclass of $\mathcal{D}$. Show that $\boldsymbol{Fr}_{\mathcal{C}}(X)$ is a homomorphic image of $\boldsymbol{Fr}_{\mathcal{C}}(X)$.

- $\mathbb{Z}^k$ is a free commutative group over $\{(1,0,\ldots,0),\ldots,\{(0,\ldots,0,1)\}$;
- This is a free group of rank 2.



source: https://tex.stackexchange.com/questions/222881/cayley-graph-of-free-group-in-tikz

- The ring of polynomials with integer coefficients and with variables in X is free in the class of commutative rings.

$$4x^2 + 3xy + z^3 - 7$$

Fact

Free algebra in a class, if exists, is unique up to isomorphism.

Proof.

Let $\boldsymbol{Fr}(X)$ and $\boldsymbol{Fr}'(X)$, with the embedding e, e', be two free algebras over the same set X in a class $\mathcal{C}$.
By the universal property used twice, there are unique homomorphisms $\bar{e}, \bar{e'}$ such that

$$\begin{tikzcd}
 & X \arrow[dl, "e"'] \arrow[dr, "e'"] & \\
\boldsymbol{Fr}(X) \arrow[loop left, dashed, "id_{\boldsymbol{Fr}(X)}"] \arrow[rr, dashed, shift left, "\bar{e'}"] & & \boldsymbol{Fr}'(X) \arrow[ll, dashed, shift left, "\bar{e}"] \arrow[loop right, dashed, "id_{\boldsymbol{Fr}'(X)}"]
\end{tikzcd}$$

By the uniqueness of a homomomorphism in the universal property,

$$\bar{e} \circ \bar{e'} = id_{\boldsymbol{Fr}(X)} \quad \text{and} \quad \bar{e'} \circ \bar{e} = id_{\boldsymbol{Fr}'(X)}.$$

Definitions and characterizations

Let $\mathcal{K}$ be a class of algebras.

- $\mathcal{K}$ is a **quasivariety** if it is axiomatizable by a set of quasi-identities:

$$\forall \bar{x}\,(\varphi_1 = \psi_1 \mathbin{⩕} \cdots \mathbin{⩕} \varphi_n = \psi_n) \to \varphi = \psi$$

  Equiv., it is closed under S, $\mathbf{P}$ (products), and P_U (Mal'cev).

- $\mathcal{K}$ is a **variety** if it is axiomatizable by a set of identities:

$$\forall \bar{x}\, \varphi = \psi.$$

  Equiv., it is closed under H (homomorphic images), S, and P (Birkhoff).

Exercise

Let X be a set and $\mathcal{C}$ be a class of algebras containing at least one algebra with at least two elements. Then a free algebra $\boldsymbol{Fr}(X)$ for $\boldsymbol{C}$ is free for $V(\mathcal{C})$.

Fact

Let $\mathcal{C}$ be a set contains at least one algebra with at least two element. Then a free algebra $\boldsymbol{Fr}(X)$ in $V(\mathcal{C})$ exists and is (isomorphic to) the subalgebra of

$$\prod_{\substack{\boldsymbol{A}\in\mathcal{C}\\ v: X\to A}} \boldsymbol{A}$$

generated by $e(X)$, where $e(x)\colon (\boldsymbol{A}, v\colon X \to A) \mapsto v(x)$.

Proof.
Let $v\colon X \to A$, $\boldsymbol{A} \in \mathcal{C}$. Then define $\bar{v}\colon \boldsymbol{Fr}(X) \to A$ as the projection on $(\boldsymbol{A}, v)$'th coordinate. Uniqueness of $\bar{v}$ follows from the fact that $\boldsymbol{Fr}(X)$ is generated by $e(X)$.
Finally, $\boldsymbol{Fr}(X)$ is free in $V(\mathcal{C})$ by the Exercise on the previous slide.

Fact

Let $\mathcal{C}$ be a set contains at least one algebra with at least two element. Then a free algebra $\boldsymbol{Fr}(X)$ in $V(\mathcal{C})$ exists and is (isomorphic to) the subalgebra of

$$\prod_{\substack{\boldsymbol{A}\in\mathcal{C} \\ v:X\to A}} \boldsymbol{A}$$

generated by $e(X)$, where $e(x)\colon(\boldsymbol{A}, v\colon X \to A) \mapsto v(x)$.

Exercise

Ok, how about the case when $\mathcal{C}$ is just a class, not a set, of algebras?

size	logics	SC logics	ASC logics
Normal extensions of D=K⊕ $\Diamond$ T			
1	1	100 %	100 %
2	5	40.0 %	100 %
3	62	24.2 %	85.5 %
4	2214	12.1 %	74.4 %
5	244134	7.9 %	76.4 %
6	87722854	5.7 %	83.3 %

Table: Normal extensions of K

Fact **This is important!**

Let $\mathcal{C}$ be a class of algebras and $\boldsymbol{Fr}_{\mathcal{C}}(\bar{x})$ be free in $\mathcal{C}$. Let $\varphi, \psi \in \boldsymbol{Form}(\bar{x})$. Then

$$\mathcal{C} \models \forall \bar{x}\ \varphi(\bar{x}) = \psi(\bar{x}) \quad \text{iff} \quad \boldsymbol{Fr}_{\mathcal{C}}(\bar{x}) \models \varphi(\bar{x}) = \psi(\bar{x}).$$

Exercise

Prove the above fact.

Exercise

Verify that

- $\mathbb{N}^k$ is a free commutative monoid over $\{(1, 0, \ldots, 0), \ldots, \{(0, \ldots, 0, 1)\}$;
- $(\{0, 1\}^n, \vee)$ is a free semilattice over $\{(1, 0, \ldots, 0), \ldots, \{(0, \ldots, 0, 1)\}$.

Theorem

Let $\vdash$ be an algebraizable consequence relation with a quasivariety $\mathcal{Q}$ as its algebraic counterpart. Let a q quasi-identity be the translation of a rule, or vice versa.
Then Γ/δ is admissible in $\vdash$ iff $\boldsymbol{Fr}_{\mathcal{Q}} \models q$.

Proof with translation $Tr(\varphi) = (\varphi = 1)$.
Recall that $\vdash \rho$ iff $\mathcal{Q} \models \forall \bar{x}\ \rho = 1$. TFAE:

- Γ/δ is admissible;
- for every subst. s, $(\forall \gamma \in \Gamma, \vdash s(\gamma))$ yields $\vdash s(\delta)$;
- for every s, $(\forall \gamma \in \Gamma, \mathcal{Q} \models \forall \bar{x}\ s(\gamma) = 1)$ yields $\mathcal{Q} \models \forall \bar{x}\ s(\delta) = 1$;
- for every s, $(\forall \gamma \in \Gamma, \boldsymbol{Fr}_{\mathcal{Q}} \models s(\gamma) = 1)$ yields $\boldsymbol{Fr}_{\mathcal{Q}} \models s(\delta) = 1$;
- $\boldsymbol{Fr}_{\mathcal{Q}} \models \forall \bar{x}\ \bigwedge_{\gamma} \gamma = 1 \Rightarrow \delta = 1$.

Definition (Bergman)

A quasi-identity q in **admissible in** a quasivariety $\mathcal{Q}$ if $\boldsymbol{Fr}_Q \models q$. A quasivariety $\mathcal{Q}$ is structurally complete if every admissible in $\mathcal{Q}$ quasi-identity holds in $\mathcal{Q}$ (we may say, it is **derivable in** $\mathcal{Q}$).

Corollary

Let $\vdash$ be an algebraizable consequence relation. Let the quasivariety $\mathcal{Q}$ be its algebraic counterpart. Then $\vdash$ is structurally complete iff $\mathcal{Q}$ is structurally complete.

Definition (Bergman)

A quasi-identity q in **admissible in** a quasivariety $\mathcal{Q}$ if $\boldsymbol{Fr}_Q \models q$.
A quasivariety $\mathcal{Q}$ is structurally complete if every admissible in $\mathcal{Q}$ quasi-identity holds in $\mathcal{Q}$.

Theorem (Bergman)

Let $\mathcal{Q}$ be a quasivariety. Then TFAE:

- $\mathcal{Q}$ is structurally complete;
- $\mathcal{Q} = Q(\boldsymbol{Fr}_{\mathcal{Q}}) = SPP_U(\boldsymbol{Fr}_{\mathcal{Q}})$;
- every Q-subdirrectly irreducible algebra is in $Q(\boldsymbol{Fr}_{\mathcal{Q}})$;

Proof. The first equivalence is just a reformulation to the language of classes. The second equivalence follows from Birkhoff-Mal'cev's subdirect representation theorem.

Theorem (Bergman refined)

Let $\mathcal{Q}$ be a quasivariety. Then TFAE:

1. $\mathcal{Q}$ is structurally complete;
2. $\mathcal{Q} = Q(\boldsymbol{Fr}_{\mathcal{Q}}) = SPP_U(\boldsymbol{Fr}_{\mathcal{Q}})$;
3. every Q-subdirrectly irreducible algebra is in $Q(\boldsymbol{Fr}_{\mathcal{Q}})$;
4. every Q-subdirrectly irreducible algebra is in $SP_U(\boldsymbol{Fr}_{\mathcal{Q}})$.
5. every finitely generated Q-subdirrectly irreducible algebra is in $SP_U(\boldsymbol{Fr}_{\mathcal{Q}})$.

Proof.
3⟺4 follows by the definition of subdirectly irreducible algebras.
4⟺5 follows from the fact that two quasivarieties coincide iff they share the same finitely generated algebras.

Exercise

Justify the fact used in in the argumentation for 4⟺5.

- The variety of vector spaces over a fixed field $\mathbb{F}$ is structurally complete. The only subdirectly irreducible algebra here is $(\mathbb{F}, +, \mathbb{F})$, which is free.
- The same holds for the variety of (bounded) distributive lattice. The only subdirectly irreducible (bounded) distributive lattice is a two element lattice and it embeds into free distributive lattice or rank 2 (it is a free distributive lattice or rank 0).
- The same applies to the variety of boolean algebras. Consequently, classical propositional calculus is structurally complete.

Exercise

Check that the variety of (commutative) groups is not structurally complete.

Fact (Bergman)

The variety of abelian groups is not structurally, but every its proper subvariety does.

A bit more difficult Exercise

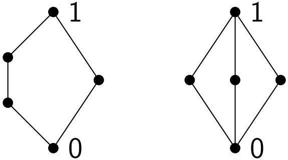
Verify the above fact. Two facts may be useful:

- Each such variety is of the form $V(\mathbb{Z}_n)$ and is definable by one identity $\forall x\ nx = 0$.
- Every finite subdirectly irreducible abelian group is of the form $\mathbb{Z}_{p^k}$, where p is prime and $p^k | n$.
- You may start with an small example for, say, $n = 12$.

Fact (Dzik, S.)

The only subdirectly irreducible structurally complete variety of bounded lattices is the variety of distributive ones.

Proof.
Let $\mathcal{V}$ be a variety containing a nondistributive bounded lattice $\boldsymbol{L}$. Then $\boldsymbol{N}_5 \in H(\boldsymbol{L}) \subseteq \mathcal{V}$ or $\boldsymbol{M}_3 \in H(\boldsymbol{L}) \subseteq \mathcal{V}$.



$\boldsymbol{N}_5$ $\boldsymbol{M}_3$

It follows that if $\mathcal{V}$ is structurally complete then one of these lattices embeds into $\boldsymbol{Fr}_{\mathcal{V}}$. Thus, 1 is join reducible in $\boldsymbol{Fr}_{\mathcal{V}}$ (in every its el. extension), i.e., $1 = \varphi(\bar{x}) \vee \psi(\bar{x})$ for some $\varphi(\bar{x}), \psi(\bar{x}) < 1$.

Fact (Dzik, S.)

The only subdirectly irreducible structurally complete variety of bounded lattices is the variety of distributive ones.

Proof cont..
In particular, 1 is join reducible in $\boldsymbol{Fr}_{\mathcal{V}}$, i.e., $1 = \varphi(\bar{x}) \vee \psi(\bar{x})$ for some $\varphi(\bar{x}), \psi(\bar{x}) < 1$.
We rebuild formulas φ and ψ to φ^* and ψ^* by replacing their subformulas in the following way

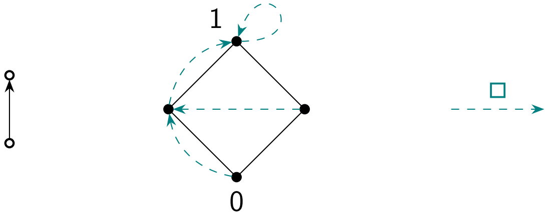
$$1 \vee \delta \mapsto 1, \quad 1 \wedge \delta \mapsto \delta.$$

Then φ^* and ψ^* are 1 or do not contain the constant 1.
Moreover, $\mathcal{V} \models \forall \bar{x}\ \varphi^*(\bar{x}) = \varphi(\bar{x}) \curlywedge \psi^*(\bar{x}) = \psi(\bar{x})$.
Thus in $\boldsymbol{Fr}_{\mathcal{V}}$, $1 = \varphi^*(\bar{x}) \vee \psi^*(\bar{x})$, $\varphi^*(\bar{x}), \psi^*(\bar{x}) < 1$, and these formulas do not contain 1.
Consequently, in $\boldsymbol{Fr}_{\mathcal{V}}$, $1 = \varphi^*(\bar{0}) \vee \psi^*(\bar{0}) = 0 \vee 0 = 0$, a contradiction.

Example in modal logic

In previous examples we dealt free algebras of infinite or arbitrarily large rank. Here is an example where only $\boldsymbol{Fr}(0)$ is needed in argumentation.

Let $\mathbb{F}$ be the depicted frame and $\boldsymbol{A} = \mathbb{F}^*$ be its dual modal algebra.



$\boldsymbol{A}$ is zero generated. Hence, for $\mathcal{V} = V(A)$, $\boldsymbol{Fr}(0) \approx \boldsymbol{A}$.
In $\mathcal{V}$ there are two subdirrectly ireducible algebras: $\boldsymbol{A}$ and $\boldsymbol{B}$ which is a two element modal algebra with $\Box 1 = 1 = \Box 0$.
By Bergman's theorem, structural completeness of $\mathcal{V}$ would yield the embedding of $\boldsymbol{B}$ into $\boldsymbol{A}$. But it does not exist.

Fact

Let $\mathcal{C}$ be a set contains at least one algebra with at least two element. Then a free algebra $\boldsymbol{Fr}(X)$ in $V(\mathcal{C})$ exists and is (isomorphic to) the subalgebra of

$$\prod_{\substack{\boldsymbol{A}\in\mathcal{C} \\ v:X\to A}} \boldsymbol{A}$$

generated by $e(X)$, where $e(x)\colon(\boldsymbol{A}, v\colon X \to A) \mapsto v(x)$.

Proof.

Let us find dual space to a free Boolean algebra $\boldsymbol{Fr}_{\mathcal{BA}}(n)$.
Let $\mathbf{2}$ be a two element boolean algebra. Let $\mathcal{BA} = V(\mathbf{2})$ be the variety of boolean algebras. Then

$$\boldsymbol{Fr}_{\mathcal{BA}}(n) \leqslant \prod_{v : n \to 2} \mathbf{2}$$

is a subalgebra generated by $e(x) \colon v \mapsto v(x)$.
As the dual space to $(\mathbf{2}^{2^n})_*$ we may take 2^n-element set set $T = \{t_v : v \colon n \to 2\}$. The generators $e(x)$, $x \in n$, correspond to subsets $\mathrm{val}(x) = \{t_v : v(x) = 1\} = \{t_v : t_v \Vdash x\}$. The sought subalgebra of $\mathbf{2}^{2^n}$ corresponds to the quotient $T/\sim$, where
$t_v \sim t_w$ iff ($\forall x \in n$, $t_v \Vdash x$ iff $t_w \Vdash x$)
iff ($\forall x \in n$, $v(x) = 1$ iff $w(x) = 1$) iff $v = w$,
i.e., $\sim = id_T$. We conclude that $\boldsymbol{Fr}_{\mathcal{BA}}(n) \approx T^* \approx \mathbf{2}^{2^n}$.

Universal Frames

Suppose that there is a reasonable duality theory for a given quasivariety. Object dual to free algebras (of rank n) are called **universal** (of rank n).

So an 2^n-element set is a universal stone space of rank n.

We will be talking about universal models and universal frames for intermediate and modal logics. We use the notation $\mathbb{U}(X)$, or $\mathbb{U}_{\mathsf{L}}(X)$ when we want to indicate the logic L under consideration.

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We have just indicated that free algebras are subalgebras of a product generated by $e(X)$. So let us describe how to dualize generation of subalgebra.

Definition

Let $\mathbb{M} = (\mathbb{F}, \mathrm{val})$, be a Kripke model with a set of variables X. A relation $\sim \subseteq F^2$ is a **bisimulation** if provided that

(ZZP) for all $u_1, u_2, v_1 \in F$, if $u_1 \sim u_2$ and $u_1 \, R \, v_1$ then there exists $v_2 \in F$ such that $u_2 \, R \, v_2$ and $v_1 \sim v_2$ (*the zigzag property property*),

(ZZP) for all $u_1, u_2, v_1 \in F$, if $u_1 \sim u_2$ and $u_2 \, R \, v_2$ then there exists $v_1 \in F$ such that $u_1 \, R \, v_1$ and $v_1 \sim v_2$ (*the zigzag property property*),

(HaP) for all $u \in F$ and $x \in X$, $u \Vdash x$ iff $h(u) \Vdash x$ (*the harmony property*).

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Observations

- Reflexive, symmetric, transitive closure of a bisimulation is a bisimulation.
- A union of bisimulations is a bisimulation.
- Thus, there exists a largest bisimulation and is is an equivalence.

Theorem (Hennessy and Milner)

Let $\mathbb{M}$ be a finite (or, slightly more general, image finite) Kripke model with variables in X. Let $\sim$ be its largest bisimulation. Then

$$u \sim v \quad \text{iff} \quad \forall \varphi \in \mathit{Form}(X)\ (u \Vdash \varphi \text{ iff } v \Vdash \varphi).$$

.

Corollary (variant for modal models)

Let $\mathbb{M}$ be a finite model and $\sim$ be its largest bisimulation. Then, for every $u \in F$ there exists a formula φ_u such that

$$v \Vdash \varphi_u \quad \text{iff} \quad u \sim v.$$

Corollary (variant for modal models)

Let $\mathbb{M}$ be a finite model and $\sim$ be its largest bisimulation. A subset a of F is a union of $\sim$ classes iff it is of the form $\mathrm{val}(\varphi)$ for some $\varphi \in \mathit{Form}(X)$.

Corollary This is important!

Let $\mathbb{M}$ be a finite Kripke model and $\sim$ be its largest bisimulation.
Let $f\colon \mathbb{M} \to \mathbb{M}/\sim$, $f(u) = u/\sim$.
Then $f^*\colon (\mathbb{M}/\sim)^* \to \mathbb{M}^*$, $f^*(b) = \bigcup b$, is an embedding of $(\mathbb{M}/\sim)^*$ onto the subalgebra of $\mathbb{M}^*$ generated by $\mathrm{val}(x)$, $x \in X$.

Corollary **This is important!**

Let $\mathbb{M}$ be a finite Kripke model and $\sim$ be its largest bisimulation. Let

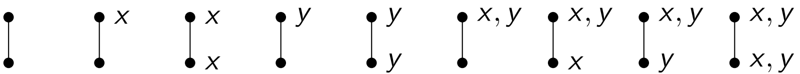
$$f\colon \mathbb{M} \to \mathbb{M}/\sim, \quad f(u) = u/\sim .$$

Then

$$f^{*}\colon (\mathbb{M}/\sim)^{*} \to \mathbb{M}^{*}, \quad f^{*}(b) = \bigcup b$$

is an embedding of $(\mathbb{M}/\sim)^{*}$ onto the subalgebra of $\mathbb{M}^{*}$ generated by $\mathrm{val}(x)$, $x \in X$.

Let us describe universal frames for the variety generated by a three element chain Heyting algebra $\mathbf{3} = \{0, m, 1\}$ over the set $X = \{x, y\}$. The dual to the algebra $\mathbf{3}^{\{0,m,1\}^{\{x,y\}}}$ with the valuation e is the Kripke model $\mathbb{M}$



The Subalgebra $\boldsymbol{Fr}(X)$ correspond to $\mathbb{U}(X) = \mathbb{M}/\!\sim$, where $\sim$ is the maximal bisimulation of $\mathbb{M}$. By Hennessy's and Milner's theorem, $t \sim s$ iff $\forall \varphi \in \boldsymbol{Form}(x, y)$ $t \vDash \varphi$ iff $s \vDash \varphi$.

Fact

Let $\boldsymbol{A}, \boldsymbol{B}$ be finite Heyting or modal algebras. Then $\boldsymbol{A}$ embeds into $\boldsymbol{B}$ iff the dual frame $\boldsymbol{A}_*$ is a p-morphic image of the dual frame $\boldsymbol{B}_*$.

Fact

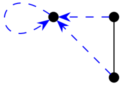
Let $\boldsymbol{S}$ be a finite Heyting or modal algebra. Then $\boldsymbol{S}$ is subdirectly irreducible iff its dual frame $\boldsymbol{S}_*$ is rooted.

Fact

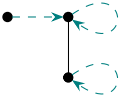
LC⊕BD2 is structurally complete.

Proof. Let $\mathcal{V} = V(\mathbf{3})$ be the variety of Heyting algebras that is an algebraic counterpart of LC⊕BD2.
$\mathcal{V}$ has two subdirrectly irreducible algebras: $\mathbf{2}$ and $\mathbf{3}$. By Bergman's theorem, it suffices to show that they are subalgebras of some $\boldsymbol{Fr}(X)$. It suffices to consider only $\boldsymbol{Fr}(x)$.

Dualizing Bergman's condition, we obtain that frames • and  are p-morphic images of a universal frame $\mathbb{U}(x)$:  .



and

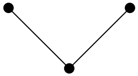


Exercise

Show that the intermediate logic LC⊕BD3, characterized by the three element chain frame, is structurally complete. Now $\mathbb{U}(x, y)$ should be considered.

Exercise

Show that the intermediate logic characterized by the fork frame is structurally complete.



There are two point in Bergman's characterization that may be difficult in applications: infinite rank of $\boldsymbol{Fr}_{\mathcal{Q}}$ and ultrapower construction. In many cases, we do not need to worry about them. A quasivariety $\mathcal{Q}$ is **locally finite** if every its finitely generated algebra is finite.

Exercise

Use the construction of free algebra to show that every finitely generated variety is locally finite.

Theorem (Bergman, more refined)

Let $\mathcal{Q}$ be a locally finite quasivariety. Then TFAE:

1. $\mathcal{Q}$ is structurally complete;
5. every finitely generated $\mathcal{Q}$-subdirrectly irreducible algebra is in $SP_U(\boldsymbol{Fr}_{\mathcal{Q}})$.
6. every finite $\mathcal{Q}$-subdirectly irreducible algebra embeds into $\boldsymbol{Fr}_{\mathcal{Q}}(k)$ for some finite k.

Theorem (Bergman, more refined)

Let $\mathcal{Q}$ be a locally finite quasivariety. Then TFAE:

1. $\mathcal{Q}$ is structurally complete;
5. every finitely generated $\mathcal{Q}$-subdirrectly irreducible algebra is in $SP_U(\boldsymbol{Fr}_{\mathcal{Q}})$.
6. every finite $\mathcal{Q}$-subdirectly irreducible algebra embeds into $\boldsymbol{Fr}_{\mathcal{Q}}(k)$ for some finite k.

Proof.
5⇐6 By local finiteness.
5⇒6 Let $\boldsymbol{S} \in \mathcal{Q}$ be finite and $\mathcal{Q}$-subdirectly irreducible. By 5, $\boldsymbol{S}$ embeds into an elementary extension $\boldsymbol{F}$ of $\boldsymbol{Fr}_{\mathcal{Q}}$. This means that $\boldsymbol{F} \models \exists \bar{x}$ (diagmam of $\boldsymbol{S}$). $\boldsymbol{F}$ and $\boldsymbol{Fr}_{\mathcal{Q}}$ are elemenarly equivalent, hence $\boldsymbol{Fr}_{\mathcal{Q}} \models \exists \bar{x}$ (diagmam of $\boldsymbol{S}$). Thus there is an embedding $i\colon \boldsymbol{S} \to \boldsymbol{Fr}_{\mathcal{Q}}$. For each $s \in S$ choose one formula φ_s such that $i(s) = \varphi_s$. Let X be the finite set of all variables occurring in any φ_s. Then $\boldsymbol{S}$ embeds into $\boldsymbol{Fr}_{\mathcal{Q}}(X)$.

Recall that LC, the intermediate logic automatized by $(\varphi \to \psi) \vee (\psi \to \varphi)$ is characterized by the set of all finite chains. Let $\mathcal{LC}$ ($\mathcal{LC}_n$) be the variety of Heyting algebras corresponding to LC (LC⊕BDn). Note that $\mathcal{LC}_n = V(n+1 \text{ element chain})$. Observe that a chain Heyting algebra $\boldsymbol{A}$ is k generated iff $|A| \leqslant k+2$. Thus $\boldsymbol{Fr}_{\mathcal{CL}}(n) = \boldsymbol{Fr}_{\mathcal{CL}_{n+1}}(n)$, and this algebra is finite.

Exercise

Verify that $\mathbb{U}_{\mathsf{LC}}(n)$ is a co-forest (all principal upsets are chains) of depth $n+1$ and with a few branches of this depth.

Exercise

Verify that every finite co-forest admits a p-morphism on any of its nonempty upset. Conclude that LC is structurally complete.

Example

In the case of LC logic we could provide argumentation on Kripke models only. Let us illustrate it on the modal logic S5 with axioms $T\colon \Box A \to A$ and 5: $\Diamond A \to \Box \Diamond A$. It is characterized by finite frames with total relations, called **clusters**. For this purpose recall:

Fact

Let $\mathbb{F}$ be a finite intuitionistic or modal frame. Then $\mathbb{F}^*$ is n-generated iff $\mathbb{F}$ admits n-variable valuation val such that the largest bisimulation of $(\mathbb{F}, \mathrm{val})$ is the identity.

Exercise

Verify that a cluster $\mathbb{C}$ such that $\mathbb{C}^*$ is n generated has at most 2^n elements. Conclude that $\mathbb{U}_{S5}(n)$ is a disjoint union of finitely many clusters with at most 2^n elements.

Exercise

Why $\mathbb{U}_{S5}(n)$, for any n, does not admits a p-morphism onto a two element cluster? Why it shows that S5 is not structurally complete?